NOTES ON LOCALLY FREE CLASS GROUPS

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ABSTRACT. A theorem of Swan states that the locally free class group of a maximal order in a central simple algebra is isomorphic to a restricted ideal class group of the center. In this article we discuss this theorem and its generalization to separable algebras for which it is more applicable to integral representations of finite groups. This is an expository article with aim to introduce the topic for non-specialists.

1. Introduction

A more precise title of this article should be "Notes on locally free class groups of orders in separable algebras over global fields". Our goal is to introduce the locally free class group of an R-order Λ in a separable K-algebra. Here R is a Dedekind domain and K is its fraction field, which we shall assume to be a global field later. We refer to Section 3 for definitions of separable K-algebras and R-orders.

The notion of locally free class groups can be defined in a more general setting. However, since results discussed here are only restricted to the case where the ground field K is a global field, we do not attempt to discuss its definition as general as it could be. Instead we shall illustrate the essential idea of this notion (see Section 2).

After illustrating the notion, we present the main theorem on locally free class groups. We then explain how to deduce a theorem of Swan [9, Theorem 1, p. 56] from the main theorem. The strong approximation theorem (SAT) plays an important role in the proof of the main theorem, and we give a short exposition of the SAT. Our another goal to take this opportunity to introduce the reader (mainly for graduate students and non-specialists) some useful tools and results in Algebraic Number Theory and show how to apply them together.

2. The cancellation law

Let us first motivate the notion of locally free class groups by the classical theorem of Steinitz. Let R be a Dedekind domain with fraction field K and we will always suppose that $R \neq K$. An *R-lattice* is a finite R-module M which does not have non-zero torsion elements, that is, M is isomorphic to a finite R-submodule in a (finite-dimensional) K-vector space. We have the following results concerning the classification of R-lattices (cf. [1, Theorem (4.13), p. 85]):

(1) Every R-lattice M is R-projective, and $M \simeq \oplus_{i=i}^{n} J_i$ for some non-zero ideals J_i of R, where n is the R-rank of M.

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(2) Two R-lattices $M=\oplus_{i=i}^n J_i$ and $M'=\oplus_{i=i}^m J_i'$ of the form in (1) are isomorphic if and only if $n=m$ and the products $J_1\cdots J_n$ and $J_1'\cdots J_n'$ are isomorphic.

From the statement (2) one can easily deduce the following result: If M and M' are two R-lattices, then we have

$$M\oplus R\simeq M'\oplus R \iff M\simeq M' \tag{2.1}$$

The property (2.1) is called the *cancellation law*.

As we learned from Algebra, a useful and easier way of studying rings to study their modules, instead of their underlying ring structures. Using this approach, the cancellation law then can be used to distinguish certain rings which share the same good properties. For example, consider the quaternion $\mathbb{Q}$-algebras $B_{p,\infty}$, which are those ramified exactly the two places $\{p,\infty\}$ of $\mathbb{Q}$, for primes p. Choose a maximal order $\Lambda(p)$ in each $B_{p,\infty}$, that is, $\Lambda(p)$ is not strictly contained in another $\mathbb{Z}$-order in $B_{p,\infty}$. Then one can show that the cancellation law for ideals of $\Lambda(p)$ holds true if and only if $p\in\{2,3,5,7,13\}$. We will also give a proof of this fact later.

Now let Λ be an (not necessarily commutative) R-algebra which is finitely generated as an R-module. The above example shows that the cancellation law for (right) projective Λ-modules need not hold. In Mathematics, we often encounter a situation that a nice property we are looking for turns out to be impossible. In that situation one usually remedies it by creating a more flexible notion so that the desired nice property remains valid in a slightly weaker setting. In the present case, one can for example consider the following weaker equivalence relation:

$$\text{Define } M\sim M' \quad \text{if } M\oplus\Lambda^r\simeq M'\oplus\Lambda^r \text{ for some integer } r\ge 0. \tag{2.2}$$

Then it follows from the definition that the cancellation law holds true for this new equivalence relation, that is, we have

$$M\oplus\Lambda\sim M'\oplus\Lambda \iff M\sim M'. \tag{2.3}$$

The modules M and M' satisfying the above property are said to be *stably isomorphic*. The reader familiar with algebraic topology would immediately find that the way of defining a "stable" notion here is similar to that in the definition of stable homotopy groups. It is also similar to that in the definitions of stable freeness and stable rationality. These are reminiscent of the definition of groups K_0 and K_1 in Algebraic K-theory using an inductive limit procedure.

3. Locally free class groups

For the rest of this article we assume that the ground field K is a global field; that is, K is a finite separable extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$. Thus, R is the ring of S-integers of K for a finite set S of places which contains all archimedean ones when K is a number field. Let A denote a finite-dimensional separable K-algebra. That is, A is a finite-dimensional semi-simple K-algebra such that the center C of A is a product of finite separable field extensions K_i of K. Recall that an *R-order* in A is an R-subring of A which is finitely generated as an R-module and generates A over K. We let Λ denote an R-order in A. A *Λ-lattice* M is a R-torsion free finitely generated Λ-module.

Example. Let G be a finite group with $\mathrm{char} K \nmid |G|$. Then the group algebra $A = KG$ is a separable K-algebra. We can see this by Maschke's Theorem (cf. [1, Theorem 3.14, p. 42]): Every finite-dimensional representation of G over K is a direct sum of irreducible representations. Then by definition KG is a semi-simple K-algebra. Applying Maschke's Theorem again to an algebraic closure $\overline{K}$ of K, we see that the algebra $\overline{K} \otimes_K KG = \overline{K}G$ is also semi-simple. Therefore, A is a separable K-algebra. Clearly, the group ring $\Lambda = RG$ is an R-order in A, and any representation M of G over R is a Λ-lattice.

For any integer $n \geq 1$, denote by $\mathrm{LF}_n(\Lambda)$ the set of isomorphism classes of locally free right Λ-modules of rank n. Two locally free right Λ-modules M and M' are said to be *stably isomorphic*, denoted by $M \sim_s M'$, if

$$M \oplus \Lambda^r \simeq M' \oplus \Lambda^r$$

as Λ-modules for some integer $r \geq 0$. The stable class of M will be denoted by $[M]_s$, while the isomorphism class is denoted by $[M]$. By a Λ-ideal we mean a Λ-lattice in A, that is, it is an R-lattice which is also a Λ-module. Let $\mathrm{Cl}(\Lambda)$ denote the set of stable classes of locally free right Λ-ideals in A. The Jordan-Zassenhaus Theorem (cf. [1, Theorem 24.1, p. 534]) states that $\mathrm{LF}_1(\Lambda)$ is a finite set, and hence so the set $\mathrm{Cl}(\Lambda)$ is. We define the group structure on $\mathrm{Cl}(\Lambda)$ as follows. Let J and J' be two locally free Λ-ideals. Define

$$[J]_s + [J']_s = [J'']_s, \tag{3.1}$$

where J'' is any locally free Λ-ideal satisfying

$$J \oplus J' = J'' \oplus \Lambda \tag{3.2}$$

as Λ-modules. Such a Λ-ideal J'' always exists and we will see this in Section 4. The following basic lemma shows that $\mathrm{Cl}(\Lambda)$ is an abelian group, called the *locally free class group* of Λ.

Lemma 1. *The finite set* $\mathrm{Cl}(\Lambda)$ *with the binary operation defined in (3.1) forms an abelian group.*

PROOF. By (3.2), the commutativity holds true. We prove the associativity. Let J_1, J_2, J_3 be three locally free ideals of Λ. Suppose we have $[J_1]_s + [J_2]_s = [J']_s$ and $[J']_s + [J_3]_s = [J'']_s$. Then

$$(J_1 \oplus J_2) \oplus J_3 \simeq \Lambda \oplus J' \oplus J_3 \simeq J'' \oplus \Lambda^2.$$

Similarly if $[J_2]_s + [J_3]_s = [G']_s$ and $[J_1]_s + [G']_s = [G'']_s$, then $J_1 \oplus (J_2 \oplus J_3) \simeq G'' \oplus \Lambda^2$. This shows $[J'']_s = [G'']_s$ and the associativity holds true. ∎

We introduce some more notations. Denote by C the center of A. One has $C = \prod_i^s K_i$ and $A = \prod_i^s A_i$, where each A_i is a central simple algebra over K_i. For any place v of K, let K_v denote the completion of K at v, and O_v the valuation ring if v is non-archimedean. We also write R_v for O_v when $v \notin S$. Let $A_v := K_v \otimes_K A$, $C_v := K_v \otimes_K A$ and $\Lambda_v := R_v \otimes_R \Lambda$ be the completions of A, C and Λ at v, respectively. By a place w of C we mean a place w of K_i for some i; that the algebra A splits (resp. is ramified) at the place w of C means that A_i splits (resp. is ramified) at the place w. Let $\widehat{R} = \prod_{v \notin S} R_v$ be the profinite completion of R, and

let $\widehat{K} = K \otimes_R \widehat{R}$ be the finite S-adele ring of K; one also writes $\mathbb{A}_K^S$ for $\widehat{K}$. Put $\widehat{A} := \widehat{K} \otimes_K A$, $\widehat{C} := \widehat{K} \otimes_K C$ and $\widehat{\Lambda} := \widehat{R} \otimes_R \Lambda = \prod_{v \notin S} \Lambda_v$.

It is a basic fact that the set $\mathrm{LF}_1(\Lambda)$ is isomorphic to the double coset space $A^\times \backslash \widehat{A}^\times / \widehat{\Lambda}^\times$. There is a natural surjective map

$$\mathrm{LF}_1(\Lambda) \to \mathrm{Cl}(\Lambda) \tag{3.3}$$

by sending $[J] \mapsto [J]_s$. Let $N_{A_i/K_i} : A_i \to K_i$ denote the reduced norm map. It induces a surjective map $N_i : \widehat{A}_i^\times \to \widehat{K}_i^\times$ because we have the surjectivity $A_i \otimes K_v \to K_i \otimes K_v$ for any finite place v of K. The reduced norm map $N : A = \prod_i A_i \to C = \prod_i K_i$ is simply defined as the product $N = (N_{A_i/K_i})_i$. Then we have a surjective map $N : \widehat{A}^\times \to \widehat{C}^\times$, and it gives rise to surjective map (again denoted by)

$$N : \mathrm{LF}_1(\Lambda) \simeq A^\times \backslash \widehat{A}^\times / \widehat{\Lambda}^\times \to N(A^\times) \backslash \widehat{C}^\times / N(\widehat{\Lambda}^\times). \tag{3.4}$$

We will see that $N(A^\times) = C_{+,A}^\times$, where $C_{+,A}^\times \subset C^\times$ consists of all elements $a \in C^\times$ with $r(a) > 0$ for all real embeddings(places) r which are ramified in A. The main theorem for the locally free class groups is as follows.

Theorem 2. *The map (3.4) factors through* $\mathrm{LF}_1(\Lambda) \to \mathrm{Cl}(\Lambda)$ *and it induces an isomorphism of finite abelian groups*

$$\nu : \mathrm{Cl}(\Lambda) \simeq \widehat{K}^\times / C_{+,A}^\times N(\widehat{\Lambda}^\times). \tag{3.5}$$

We now describe the theorem of Swan on locally free class groups. Assume that A is a central simple algebra and Λ is a maximal R-order in A. Define the ray class group $\mathrm{Cl}_A(R)$ of K by

$$\mathrm{Cl}_A(R) := I(R)/P_A(R),$$

where $I(R)$ be the ideal group of R and $P_A(R)$ be the subgroup generated by the principal ideals (a) for $a \in K_{+,A}^\times$. Here $K_{+,A}^\times \subset K^\times$ is the subgroup of $K^\times$ defined as above. In terms of the adelic language, the group $\mathrm{Cl}_A(R)$ is nothing but the group $\widehat{K}^\times / K_{+,A}^\times \widehat{R}^\times$.

Theorem 3 (Swan)**.** *Let K be a global field and R the ring of S-integers of K for a finite set S of places containing all archimedean ones. Let A be a central simple algebra and Λ a maximal R-order in A. Then theres is an isomorphism of finite abelian groups* $\mathrm{Cl}(\Lambda) \simeq \mathrm{Cl}_A(R)$.

To see Theorem 3 is an immediate consequence of Theorem 2, we just need to check that $N(\Lambda_v^\times) = R_v^\times$ for $v \notin S$ (Λ_v here is a maximal R_v-order). There exists a maximal subfield $E \subset A_v$ which is unramified over K_v. Since any two maximal orders in A_v are conjugate, Λ_v contains a copy of the ring of integers O_E of E. As E is unramified over K_v, the successive approximation shows that $N_{E/K_v}(O_E^\times) = R_v^\times$. It follows that $N(\Lambda_v^\times) = R_v^\times$.

Proposition 4. *Let $B_{p,\infty}$ and $\Lambda(p)$ for primes p be as in Section 2. Then the cancellation law for ideals of $\Lambda(p)$ holds true if and only if $p \in \{2, 3, 5, 7, 13\}$.*

Proof. The cancellation law holds if and only if the map $\mathrm{LF}_1(\Lambda(p)) \to \mathrm{Cl}(\Lambda(p))$ is a bijection. By Swan's theorem, the locally free class group $\mathrm{Cl}(\Lambda(p)) \simeq \mathrm{Cl}_{B_{p,\infty}}(\mathbb{Z})$

is trivial. Thus, the cancellation law holds if and only if the class number $h(\Lambda(p)) = |\mathrm{LF}_1(\Lambda(p))|$ is one. On the other hand we have the class number formula [3]

$$h(\Lambda(p)) = \frac{p-1}{12} + \frac{1}{3}\left(1 - \left(\frac{-3}{p}\right)\right) + \frac{1}{4}\left(1 - \left(\frac{-4}{p}\right)\right), \tag{3.6}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. From this one easily sees that $h(\Lambda(p)) = 1$ if and only if $p \in \{2, 3, 5, 7, 13\}$. ∎

For the rest of this section we give a proof of the following basic fact.

Lemma 5. *Let A is a separable K-algebra and C its center. Then $N(A^\times) = C^\times_{+,A}$.*

Proof. Since $A = \prod_i A_i$ and $C^\times_{+,A} = \prod_i K^\times_{i,+,A_i}$, it suffices to show $N(A^\times) = K^\times_{+,A}$ for any central simple K-algebra A. We can use the Hasse-Schilling norm theorem (the local-global principle for the reduced norm map) to describe $N(A^\times)$:

$$N(A^\times) = \{x \in K^\times; x \in N(A_v^\times)\ \forall\, v\};$$

see [8, (32.9) Theorem, p. 275] and [8, (32.20) Theorem, p. 280]). Clearly $N(A_v^\times) = K_v^\times$ when v is complex, non-archimedean, or a real split place for A. It remains to show that if v is a real ramified place for A, then one has $v(a) > 0$ if and only if $a \in N(A_v^\times)$.

Lemma 6. *Let $\mathbb{H}$ be the real Hamilton quaternion and $n \in \mathbb{N}$. Then $N(\mathrm{GL}_n(\mathbb{H})) = \mathbb{R}_+$.*

Proof. We give two proofs of this result. One is topological and the other one is algebraic. As $\mathbb{R}_+ = N(\mathbb{H}^\times) \subset N(\mathrm{GL}_n(\mathbb{H}))$, it suffices to show $N(\mathrm{GL}_n(\mathbb{H})) \subset \mathbb{R}_+$.

(1) The set $\mathrm{GL}_n(\mathbb{H})^{ss} \subset \mathrm{GL}_n(\mathbb{H})$ of semi-simple elements is open and dense in the classical topology. By continuity it suffices to show $N(x) > 0$ for any $x \in \mathrm{GL}_n(\mathbb{H})^{ss}$. Since such x is contained in a maximal commutative semi-simple subalgebra, which is isomorphic to $\mathbb{C}^n$, we have $N(x) > 0$.

(2) The algebraic proof relies on the existence of the Dieudonné (non-commutative) determinant (cf. [1, p. 165]). Suppose that D is a central division algebra over any field K. There is a group homomorphism (called the Dieudonné determinant)

$$\det : \mathrm{GL}_n(D) \to D^\#,$$

where $D^\# = D^\times/[D^\times, D^\times]$. The reduced norm map $N : D^\times \to K^\times$ gives rise to a map $\mathrm{nr} : D^\# \to K^\times$. The reduced norm map $N : \mathrm{GL}_n(D) \to K^\times$ can be also described as

$$N(a) = \mathrm{nr}(\det a). \quad \forall\, a \in \mathrm{GL}_n(D).$$

It follows that $N(\mathrm{GL}_n(D)) \subset N(D^\times)$. Particularly $N(\mathrm{GL}_n(\mathbb{H})) \subset N(\mathbb{H}^\times) = \mathbb{R}_+$. ∎

Remark 7. One can show a slightly stronger result that the Lie group $\mathrm{GL}_n(\mathbb{H})$ is connected. Let G_1 be the kernel of the reduced norm map $N : \mathrm{GL}_n(\mathbb{H}) \to \mathbb{R}^\times$. Then $G_1 = \underline{G}_1(\mathbb{R})$ for a connected, semi-simple and simply connected algebraic $\mathbb{R}$-group $\underline{G}_1$, and hence that G_1 is connected. Then the fibers of the reduced norm map N are all connected as they are principal homogeneous spaces under G_1. We just showed that the image of the map N is also connected (Lemma 6). Thus, $\mathrm{GL}_n(\mathbb{H})$ is connected.

4. Proof of Theorem 2

For any integer $n \geq 1$ and any ring L not necessarily commutative, let $\mathrm{Mat}_n(L)$ denote the matrix ring over L and let $\mathrm{GL}_n(L)$ denote the group of units in $\mathrm{Mat}_n(L)$. Let $N_n : \mathrm{Mat}_n(A) \to C$ be the reduced norm map, which induces a surjective homomorphism $N_n : \mathrm{GL}_n(\widehat{A}) \to \widehat{C}^\times$. For any integer $r \geq 1$, let $I_r \in \mathrm{Mat}_r(\mathbb{Z})$ be the identity matrix. Let $\varphi_r : \mathrm{GL}_n \to \mathrm{GL}_{n+r}$ be the morphism of algebraic groups which sends

$$a \mapsto \varphi_r(a) = \begin{pmatrix} a & \\ & I_r \end{pmatrix}.$$

Clearly any locally free right Λ-module M of rank n is isomorphic to a Λ-submodule in A^n. Therefore, the set $\mathrm{LF}_n(\Lambda)$ is in bijection with the set of global equivalence classes of the genus of the standard lattice Λ^n in A^n. The latter is naturally isomorphic to $\mathrm{GL}_n(A)\backslash \mathrm{GL}_n(\widehat{A})/ \mathrm{GL}_n(\widehat{\Lambda})$. If $n \geq 2$, then it follows from the strong approximation theorem (see Kneser [6] and Prasad [7], also see Theorem 10) that the induced map

$$N_n : \mathrm{GL}_n(A)\backslash \mathrm{GL}_n(\widehat{A})/ \mathrm{GL}_n(\widehat{\Lambda}) \xrightarrow{\sim} \widehat{C}^\times/N_n(\mathrm{GL}_n(A))N_n(\mathrm{GL}_n(\widehat{\Lambda})) \tag{4.1}$$

is a bijection.

Lemma 8. *We have*

$$\widehat{C}^\times/N_n(\mathrm{GL}_n(A))N_n(\mathrm{GL}_n(\widehat{\Lambda})) = \widehat{C}^\times/N(A^\times)N(\widehat{\Lambda}) = \widehat{C}^\times/C^\times_{+,A}N(\widehat{\Lambda}). \tag{4.2}$$

Proof. We have seen in Lemma 5 that $N_n(\mathrm{GL}_n(A)) = N(A^\times) = C^\times_{+,A}$. We now prove $N_n(\mathrm{GL}_n(\Lambda_v)) = N(\Lambda_v^\times)$ for $v \notin S$ since the statement is local. The group $\mathrm{GL}_n(\Lambda_v)$ contains as a subgroup the group $E_n(\Lambda_v)$ of elementary matrices with values in Λ_v. Since Λ_v is semi-local, we have a result of Bass [10, Proposition 8.5] that $\mathrm{GL}_n(\Lambda_v)$ is generated by the subgroup $E_n(\Lambda_v)$ and the image $\varphi_{n-1}(\mathrm{GL}_1(\Lambda_v))$. Since $E_n(\Lambda_v)$ is contained in the kernel of N, we have $N_n(\mathrm{GL}_n(\Lambda_v)) = N_n(\varphi_{n-1}(\Lambda_v^\times)) = N(\Lambda_v^\times)$. ∎

For any integer $r \geq 1$, we say two locally free right Λ-ideals J and J' are *r-stably isomorphic* if $J \oplus \Lambda^r \simeq J' \oplus \Lambda^r$ as Λ-modules. Let $\hat{c} \in \widehat{A}^\times$ be an element such that $\hat{c}\Lambda = J$; we have $\varphi_r(\hat{c})\Lambda^{r+1} = J \oplus \Lambda^r$.

The morphism φ_r induces the following commutative diagram:

$$\begin{array}{ccc} A^\times\backslash\widehat{A}^\times/\widehat{\Lambda}^\times & \xrightarrow{\varphi_r} & \mathrm{GL}_{r+1}(A)\backslash \mathrm{GL}_{r+1}(\widehat{A})/ \mathrm{GL}_{r+1}(\widehat{\Lambda}) \\ {\scriptstyle N}\big\downarrow & & \big\downarrow{\scriptstyle N_{r+1}} \\ \widehat{C}^\times/C^\times_{+,A}N(\widehat{\Lambda}^\times) & \xrightarrow{\mathrm{id}} & \widehat{C}^\times/C^\times_{+,A}N(\widehat{\Lambda}^\times), \end{array} \tag{4.3}$$

where the reduced norm map N_{r+1} is known be a bijection. Two isomorphism classes $[J]$ and $[J']$ in $A^\times\backslash\widehat{A}^\times/\widehat{\Lambda}^\times$ are r-stably isomorphic if and only if $\varphi_r([J]) = \varphi_r([J'])$. As N_{r+1} is an isomorphism, this is equivalent to $N([J]) = N([J'])$. The latter condition is independent of r. Therefore, we conclude the following statement.

Lemma 9. *Let J and J' be two locally free right Λ-ideals. The following statements are equivalent.*

1. *J and J' are stably isomorphic.*
2. *J and J' are r-stably isomorphic for some $r \geq 1$.*

(3) *J and J' are r-stably isomorphic for all $r \geq 1$.*
(4) *One has $N([J]) = N([J'])$ in $\widehat{K}^\times / C^\times_{+,A} N(\widehat{\Lambda}^\times)$.*

Thus, the reduced norm map N induces an isomorphism

$$\nu : \mathrm{Cl}(\Lambda) \simeq \widehat{C}^\times / C^\times_{+,A} N(\widehat{\Lambda}^\times). \tag{4.4}$$

We now check that ν is a group homomorphism. Let J and J' be two locally free Λ-ideals. Let $\hat{c}$ and $\hat{c}'$ be elements in $\hat{A}^\times$ such that $\hat{c}\Lambda = J$ and $\hat{c}'\Lambda = J'$. Put $J'' := \hat{c}\hat{c}'\Lambda$. We claim that

(a) $J \oplus J' \simeq J'' \oplus \Lambda$ as Λ-modules;
(b) $\nu([J]_s)\nu([J']_s) = \nu([J'']_s)$.

Statement (a) follows from

$$\begin{bmatrix} \hat{c}\hat{c}' & 0 \\ 0 & 1 \end{bmatrix} \cdot \Lambda^2 = J'' \oplus \Lambda, \quad \text{and} \quad N_2\left(\begin{bmatrix} \hat{c}\hat{c}' & 0 \\ 0 & 1 \end{bmatrix}\right) = N_2\left(\begin{bmatrix} \hat{c} & 0 \\ 0 & \hat{c}' \end{bmatrix}\right)$$

in $\widehat{C}^\times / C^\times_{+,A} N(\widehat{\Lambda}^\times)$. Statement (b) follows from

$$\nu([J]_s)\nu([J']_s) = N([\hat{c}])N([\hat{c}']) = N([\hat{c}\hat{c}']) = \nu([J'']_s).$$

This completes the proof of Theorem 2.

5. Strong approximation and remarks

In this supplementary section we give a short exposition of the strong approximation theorem and explain how (4.1) follows immediately from this. We keep the notations of Section 3. In particular K denotes a global field and S is a nonempty finite set of places of K.

Theorem 10 (The strong approximation theorem)**.** *Let G be a connected, semisimple and simply connected algebraic group over K. Suppose that*

($*$) *G does not contain any K-simple factor H such that the topological group $H_S := \prod_{v \in S} H(K_v)$ is compact.*

Then the group $G(K)$ is dense in $G(\mathbb{A}_K^S)$.

Proof. See Kneser [6] when K is a number field and Prasad [7] when K is a global function field. The results were proved upon the Hasse principle, i.e. the map

$$H^1(K, G) \to \prod_v H^1(K_v, G)$$

is injective. The Hasse principle was known to hold for any simply-connected group at that time except possibly for those of type E_8. The last case of type E_8 was finally completed by Chernousov in 1989. ∎

The strong approximation theorem is a strong version of "class number one" result.

Corollary 11. *Let G be as in Theorem 10 satisfying the condition ($*$) and assume that S contains all archimedean places of K. Then for any open compact subgroup $U \subset G(\mathbb{A}_K^S)$, the double coset space $G(K)\backslash G(\mathbb{A}_K^S)/U$ consists of a single element.*

Let A, C and R be as in Section 3. Now we let G and $\underline{C}^\times$ denote the algebraic groups K associated to the multiplicative groups of A and C, respectively. For any commutative K-algebra L, one has

$$G(L) = (A \otimes_K L)^\times, \quad \underline{C}^\times(L) = (C \otimes_K L)^\times.$$

We denote again by $N : G \to \underline{C}^\times$ the homomorphism of algebraic K-groups induced by the reduced norm map $N : A \to C$, and let $G_1 = \ker N$ denote reduced norm-one subgroup. It is easy to see that the base change $G_1 \otimes \overline{K}$ is a finite product of simple groups of the form SL_m, and hence G_1 is semi-simple and simply connected.

Recall that A is said to satisfy the *Eichler condition with respect to* S, if for any simple factor A_i of A there is one place w of the center K_i over some place v in S such that the completion $A_{i,w}$ at w is not a *division* $K_{i,w}$-algebra. Another way to rephrase the last condition for A_i is that the kernel of the reduced norm map

$$N_{A_i/A_i} : \prod_{v\in S} (A_i \otimes K_v)^\times \to \prod_{v \in S} (K_i \otimes K_v)^\times$$

is not compact. In other words, the algebra A satisfies the Eichler condition with respect to S (also denote A=Eichler/R, where R is the ring of S-integers of K) if and only if the reduced norm-one subgroup G_1 satisfies the condition $(*)$ in Theorem 10. In particular, we have the following special case of Theorem 10 for G_1.

Theorem 12. *Let A be a separable K-algebra and G_1 the associated reduced norm-one subgroup defined as above. If A satisfies the Eichler condition with respect to S, then $G_1(K)$ is dense in $G_1(\mathbb{A}_K^S)$.*

Theorem 12 is what we use in the proof of Theorem 2. When K is a number field, this is the first case of the strong approximation theorem, proved by Eichler [4]. Swan [11] gives a more elementary proof of this theorem.

Suppose that A=Eichler/R, and let U be an open compact subgroup of $G(\mathbb{A}_K^S) = \widehat{A}^\times$. We want to show that the induced surjective map

$$N : G(K)\backslash G(\mathbb{A}_K^S)/U \to N(G(K))\backslash \widehat{C}^\times/N(U) \tag{5.1}$$

is injective. Let $\hat{c} \in \widehat{C}^\times$ be an element and $\hat{g} \in \widehat{A}^\times$ with $N(\hat{g}) = \hat{c}$. Then the fiber of the class $[\hat{c}]$ is

$$N^{-1}([\hat{c}]) = G(K)\backslash G(K)G_1(\mathbb{A}_K^S)\hat{g}U/U. \tag{5.2}$$

If $x_1, x_2 \in G_1(A_K^S)$ be two elements, then

$$G(K)x_1\hat{g}U = G(K)x_2\hat{g}U \iff G_1(K)x_1(\hat{g}U\hat{g}^{-1}) = G_1(K)x_2(\hat{g}U\hat{g}^{-1}). \tag{5.3}$$

Thus, we get a surjective map

$$G_1(K)\backslash G_1(\mathbb{A}_K^S)/G_1(\mathbb{A}_K^S)\cap\hat{g}U\hat{g}^{-1} \to G(K)\backslash G(K)G_1(\mathbb{A}_K^S)\hat{g}U/U = N^{-1}([\hat{c}]). \tag{5.4}$$

As we know the source of (5.4) consists of one element (Corollary 11), the fiber $N^{-1}([c])$ also consists of one element. This shows that (4.1) (or (5.1)) is a bijection.

We end with this article by a few remarks. Theorem 3 was first proved by Swan [9, Theorem 1, p. 56] when K is a number field. Fröhlich [5, Theorem 2, p. 118] gave another proof of Swan's theorem using the ideles. The proof given here is the same as that of Fröhlich and of Swan; all uses Theorem 12. The statement for Swan's theorem over global fields (Theorem 3) can be found in Curtis-Reiner [2, Theorem

(49.32), p. 233] and Reiner [8, (35.14) Theorem p. 313]. Note that in Reiner [8, (35.14) Theorem p. 313] there is an assumption of A=Eichler/R when K is a global function field, but that is superfluous. Notice that Prasad's theorem, though for most general cases, came a few years after Reiner wrote his book *Maximal Orders* (published in 1975). This could be the reason why the result [8, (35.14) Theorem p. 313] is limited to those satisfying the Eichler condition in the function field case.

The updated version of Swan's Theorem (Theorem 3) is then presented in the later books by Curtis and Reiner. They also give a more general variant (Theorem 2); see [2, (49.17) Theorem, p. 225]. The proof of Theorem 2 given in Curtis-Reiner [2] is different from the original proof of Swan and Frölich; it is proved based on Algebraic K-theory. This of course brings in more insights to the topic. Nevertheless, the original proof may be more accessible for non-specialists as it is much shorter and also conceptual. A very nice exposition for the proof of Theorem 12 can be found in Section 51 of Curtis-Reiner [2], which follows Swan [11]. The paper [11] contains some minor errors; see [12, Appendix A] for corrections and improvements.

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